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Families of non-IRUP instances of the one-dimensional cutting stock problem

Jürgen Rietz, Guntram Scheithauer*, Johannes Terno[1]

Institute for Numerical Mathematics, Dresden University of Technology, Mommsenstrasse 13, D-01069 Dresden, Germany

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Abstract

In case of the one-dimensional cutting stock problem (CSP) one can observe for any instance a very small gap between the integer optimal value and the continuous relaxation bound. These observations have initiated a series of investigations. An instance possesses the integer round-up property (IRUP) if its gap is smaller than 1. In the last 15 years, some few instances of the CSP were published possessing a gap greater than 1.

In this paper, various families of non-IRUP instances are presented and methods to construct such instances are given, showing in this way, there exist much more non-equivalent non-IRUP instances as computational experiments with randomly generated instances suggest. Especially, an instance with gap equal to $\frac{10}{9}$ is obtained. Furthermore, an equivalence relation for instances of the CSP is considered to become independent from the real size parameters. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The *one-dimensional cutting stock problem* (CSP) is as follows: given an unlimited number of pieces of identical stock material of length L (e.g. wooden length, paper reels, etc.) the task is to cut b_i pieces of length ℓ_i for $i \in I = \{1, \ldots, m\}$ while minimizing the number of stock material pieces needed.

* Corresponding author. Tel.: +49-351-463-32002; fax: +49-351-463-34268.
E-mail address: scheit@math.tu-dresden.de (G. Scheithauer).

[1] Prof. Dr. Johannes Terno passed away on May 10, 2000. We lose with him a serious researcher in Discrete Optimization and Cutting & Packing.

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Throughout this paper the abbreviation $E{:=}(m;\ L;\ \ell;\ b)$ is used for an instance of the CSP with $\ell = (\ell_1,\dots,\ell_m)^{\mathrm{T}}$ and $b = (b_1,\dots,b_m)^{\mathrm{T}}$. Without loss of generality $L \geqslant \ell_1 > \cdots > \ell_m > 0$ and $b_i > 0$ for $i \in I$ are assumed.

The classical solution approach is due to Gilmore and Gomory [7]. A non-negative integer vector $a=(a_1,\dots,a_m)^{\mathrm{T}}$ (for short: $a \in \mathbb{Z}_+^m$) is called a (feasible) *cutting pattern of* E if $\ell^{\mathrm{T}} a \leqslant L$. A cutting pattern (shortly: pattern) a is said to be *maximal* if $\ell^{\mathrm{T}} a + \ell_m > L$; a is *proper* if $a \leqslant b$, i.e. $a_i \leqslant b_i,\ i \in I$. Let $J = \{1,\dots,n\}$ denote the index set of all maximal pattern $a^j = (a_{1j},\dots,a_{mj})^{\mathrm{T}}$ of E. If the integer variable x_j denotes the times pattern a^j is used, the CSP can be modelled as follows:

$$z = \sum_{j \in J} x_j \to \min \quad \text{s.t.} \quad \sum_{j \in J} a_{ij} x_j \geqslant b_i, \quad i \in I,\ x_j \in \mathbb{Z}_+,\ j \in J. \tag{1}$$

The CSP is also known as a *high multiplicity version* of the Bin Packing Problem [8] which belongs to the NP-hard problems [5]. The common heuristic solution approach consists of solving the corresponding *continuous* (LP) *relaxation*

$$z = \sum_{j \in J} x_j \to \min \quad \text{s.t.} \quad \sum_{j \in J} a_{ij} x_j \geqslant b_i, \quad i \in I,\ x_j \in \mathbb{R}_+,\ j \in J, \tag{2}$$

where $\mathbb{R}$ is the set of real numbers. Then, based on an optimal solution of (2), integer solutions for (1) are constructed by means of suitable heuristics (cf. [7,15]).

Let $z^* = z^*(E)$ and $z_c = z_c(E)$ denote the optimal values of (1) and (2) for an instance E, respectively. Practical experience and many computational tests have shown (cf. [15,20]) surprisingly there is only a small gap $\Delta(E){:=}z^*(E) - z_c(E)$ for any instance E.

These observations have initiated a number of investigations. A set $\mathscr{P}$ of instances E has the *integer round-up property* (IRUP) if $\Delta(E) < 1$ for all $E \in \mathscr{P}$ [1]. An instance E with $\Delta(E) \geqslant 1$ is called *non-IRUP instance* in the following. It is well known that the CSP does not possess the IRUP. Note, it is NP-hard to determine whether or not a given instance of the CSP has the IRUP (cf. [9]). In [9] a first counter-example was given with gap equal to 1. In the last decade instances were found having a gap larger then 1 [4,17]. Since the gaps of these instances are less than 2, the *modified integer round-up property* (MIRUP) was defined in [16]: a set $\mathscr{P}$ of instances E possesses the MIRUP if $\Delta(E) < 2$ for all $E \in \mathscr{P}$. It is conjectured in [17] that the one-dimensional CSP possesses the MIRUP.

In case an instance E possessing the IRUP has to be solved, the LP relaxation bound $\lceil z_c(E) \rceil$ yields the optimal value of problem (1). But, if E is a non-IRUP instance the verification of optimality of an integer solution found by a heuristic is much more expensive as a consequence of the NP-hardness of the CSP.

Since in numerical tests non-IRUP instances occur relatively rarely, the impression could arise that there exist only a very limited number of non-IRUP instances. But in this paper, families are presented with an infinite number of non-equivalent non-IRUP instances. These investigations are helpful for developing and testing of exact solution approaches for the one-dimensional CSP. Such algorithms (branch-and-bound algorithms are presented e.g. in [15,19] for the CSP, in [10] for the bin packing problem, and in [2] for a scheduling problem with identical machines; a cutting-plane algorithm is proposed in [18]) typically show computational difficulties with respect to rapidly growing running time when they are applied to non-IRUP instances. Moreover, average results of computational experiments mostly do not regard hard CSP instances when the number of randomly generated instances is small per series. Consequently, exact solution approaches should especially be investigated with respect to their behaviour when applied to non-IRUP instances. In order to overcome the computational difficulties, the inherent structure of non-IRUP instances needs to be analysed in more detail.

In addition, the families of non-IRUP instances presented here support the MIRUP conjecture since the gaps do not increase for increasing number of pieces or growing sizes. Moreover, the investigations could have importance also for higher dimensional CSPs. For a comprehensive overview of recent work in connection with cutting and packing problems the reader is referred to [3].

Throughout this paper some common notations are used. Let $\mathscr{M}^*$ denote the set of instances possessing IRUP. Especially for theoretical investigations, a special case of the CSP is of interest which is named *divisible case*. An instance E belongs to the set $\mathscr{D}$ of "divisible case"-instances if L is an integer multiple of any piece length. For example, the instance

$$E_D = (3;\ 132;\ 44, 33, 12;\ 2, 3, 6) \tag{3}$$

(presented in [14]) belongs to $\mathscr{D}$ and has the gap $\Delta(E_D) = \frac{137}{132} = 1.0379$ which is the largest for $\mathscr{D}$ ever found.

In our investigations also rational sizes are allowed, i.e. $\ell \in \mathbb{Q}^m_+$ and $L \in \mathbb{Q}_+$ are assumed where $\mathbb{Q}$ is the set of rational numbers. In the most cases equivalent instances with integer sizes are considered.

In the next section an equivalence relation for instances of the CSP is introduced in order to characterize different instances in a better way and to become independent from the real sizes. Then in Section 3, families of non-IRUP instances are considered in the divisible case. Families of non-IRUP instances not restricted to the divisible case will be presented in Section 4. Constructive methods to obtain non-IRUP instances will be discussed in the fifth section followed by some concluding remarks.

2. Equivalence of cutting stock problems

Among some possibilities of defining equivalence relations for instances of CSPs here only the kind of equivalence is considered which is based on the cutting patterns. Throughout this section all input data are assumed to be integral.

2.1. Pattern equivalence

Let the instances $E = (m;\ L;\ \ell;\ b)$ and $\bar{E} = (\bar{m};\ \bar{L};\ \bar{\ell};\ \bar{b})$ be given with $m = \bar{m}$ and $b = \bar{b}$, and let $a^1, \ldots, a^n$ and $\bar{a}^1, \ldots, \bar{a}^{\bar{n}}$ denote their maximal cutting patterns, respectively. Without loss of generality the cutting patterns are assumed to be sorted lexicographically decreasing.

Definition 1. E and $\bar{E}$ are called *equivalent* (pattern-equivalent) if $n=\bar{n}$ and $a^j=\bar{a}^j$ for $j=1,\dots,n$.

Hence, any feasible pattern a^j of E is also feasible for $\bar{E}$ and vice versa. For example, the following instances $\bar{E}$ are equivalent to the instance $E=(3;\ 10;\ 5,4,3;\ b)$: $\bar{E}=(3;100;\ \ell_1,40,30;\ b)$ with $41 \leqslant \ell_1 \leqslant 50$, $\bar{E}=(3;\ 100;\ 50,\ell_2,30;\ b)$ with $34 \leqslant \ell_2 \leqslant 40$, $\bar{E}=(3;100;\ 50,40,\ell_3;b)$ with $26 \leqslant \ell_3 \leqslant 30$, $\bar{E}=(3;100+d;50+d_1,40+d_2,30+d_3;\ b)$ for all $d,\ d_i \in \mathbb{Z}_+,\ i=1,2,3$, with $2d_1+2d_2+3d_3 \leqslant d$ and $d \leqslant 9$, etc.

Let $A(E)$ denote the $m\times n$-matrix containing all maximal patterns of E. Hence, E and $\bar{E}$ are equivalent if and only if $A(E)=A(\bar{E})$. Then the class $K(E)$ of instances equivalent to E can be characterized as follows:

$$K(E):=\{\bar{E}\colon A(E)=A(\bar{E})\}.$$

Obviously, $kE:=(m;\ kL;\ k\ell;\ b)\in K(E)$ for any $k\in\mathbb{Z},\ k>0$. Furthermore, if $(m;\tilde{L};\tilde{\ell};b)\in K(E)$ and $(m;\bar{L};\bar{\ell};b)\in K(E)$ then $(m;\bar{L}+\tilde{L};\bar{\ell}+\tilde{\ell};b)\in K(E)$. For that reason, $K(E)$ can be viewed as the intersection of $\mathbb{Z}^{m+1}$ and a cone which is induced by $A(E)$.

The equivalence of instances can be characterized by a system of inequalities. Let e^i denote the ith unit vector of $\mathbb{Z}^m$.

Assertion 1. *Let $\bar{a}^1,\dots,\bar{a}^n$ denote the maximal cutting patterns of instance $\bar{E}=(m;\ \bar{L};\ \bar{\ell};b)$. Then the instance $E=(m;\ L;\ \ell;\ b)$ is equivalent to $\bar{E}$ if and only if $(\ell,L)\in\mathbb{Z}^{m+1}$ is feasible for*

$$\begin{aligned}&\ell^{\mathrm{T}}\bar{a}^j \leqslant L,\ \ \ell^{\mathrm{T}}(\bar{a}^j+e^m) \geqslant L+1, \quad j=1,\dots,n,\\ &\ell_i \geqslant \ell_{i+1}+1, \quad i=1,\dots,m-1,\ \ell_m \geqslant 1.\end{aligned} \tag{4}$$

2.2. Dominance

Let a pattern matrix $A=(a^j)_{j=1,\dots,n}$ be given. In order to get an instance of the cutting stock problem with pattern matrix A, one has to determine a solution of (4).

Because of the condition $\ell_1>\dots>\ell_m$ numerous constraints in (4) will automatically be fulfilled if other constraints are fulfilled as the following example shows. Let

$$A=(a^j)_{j=1,\dots,9}=\begin{pmatrix}2&2&1&1&1&&&&\\1&&2&1&&3&2&1&\\&1&&1&2&&1&2&3\end{pmatrix}.$$

From $\ell^{\mathrm{T}}a^1 \leqslant L$ and $\ell_1>\ell_2>\ell_3$ it follows that $\ell^{\mathrm{T}}a^j<L$ for $j=2,\dots,9$ holds. Analogously, if $\ell^{\mathrm{T}}(a^9+e^3)\geqslant L+1$ holds then $\ell^{\mathrm{T}}(a^j+e^3)\geqslant L+1$ for $j=1,\dots,8$ is fulfilled. That means, only two of the inequalities in (4) are needed.

Definition 2. The cutting pattern a *dominates* the cutting pattern $\bar{a}$ if $\ell^{\mathrm{T}}a\geqslant \ell^{\mathrm{T}}\bar{a}$ for all $\ell\in\mathbb{Z}_+^m$ fulfilling $\ell_1>\dots>\ell_m$. (for abbreviation: $a>_d\bar{a}$).

For $a\in\mathbb{Z}^m$ let

$$s_0(a):=0,\quad s_i(a):=\sum_{j=1}^{i}a_j,\quad i=1,\dots,m.$$

Assertion 2. *Let the patterns a and $\bar{a}$ be given. Then*:

$$a >_d \bar{a} \quad \Leftrightarrow \quad s_i(a) \geqslant s_i(\bar{a}), \quad i = 1, \ldots, m.$$

Proof. "$\Rightarrow$" If $s_1(a) < s_1(\bar{a})$ then $a_1 < \bar{a}_1$ and hence, $a \not>_d \bar{a}$.

Now, suppose $i \in I \setminus \{1\}$ such that $s_j(a) \geqslant s_j(\bar{a})$ for $j = 1, \ldots, i-1$ and $s_i(a) < s_i(\bar{a})$. The vector $\ell \in \mathbb{Z}_+^m$ is defined such that $\ell^{\mathrm{T}} a < \ell^{\mathrm{T}} \bar{a}$. Let $\ell_0 > m - i$ and

$$\ell_j := \begin{cases} \ell_0, & j = i, \\ \ell_0 + i - j + 1, & j = 1, \ldots, i-1, \\ m + 1 - j, & j = i+1, \ldots, m. \end{cases}$$

Then

$$\begin{aligned} \ell^{\mathrm{T}} a - \ell^{\mathrm{T}} \bar{a} &= \sum_{j<i} (\ell_0 + i - j + 1)(a_j - \bar{a}_j) + \ell_0 (a_i - \bar{a}_i) \\ &\quad + \sum_{j>i} (m + 1 - j)(a_j - \bar{a}_j) \\ &= \underbrace{(i+1)(s_{i-1}(a) - s_{i-1}(\bar{a}))}_{=:\alpha} + \ell_0 (s_i(a) - s_i(\bar{a})) \\ &\quad \underbrace{- \sum_{j<i} j(a_j - \bar{a}_j)}_{=:\beta} + \underbrace{\sum_{j>i} (m + 1 - j)(a_j - \bar{a}_j)}_{=:\gamma} \\ &= \ell_0 (s_i(a) - s_i(\bar{a})) + \alpha - \beta + \gamma. \end{aligned}$$

Since α, β and γ are independent of ℓ_0 and $s_i(a) - s_i(\bar{a}) < 0$, it follows

$$\ell^{\mathrm{T}} a - \ell^{\mathrm{T}} \bar{a} < 0$$

if ℓ_0 is sufficiently large.

"$\Leftarrow$" Now $s_i(a) \geqslant s_i(\bar{a})$ for all i and $\ell_1 > \cdots > \ell_m > 0$ is assumed. Let $\ell_{m+1} = 0$.

$$\begin{aligned} \ell^{\mathrm{T}} a - \ell^{\mathrm{T}} \bar{a} &= \sum_{j=1}^{m} \ell_j (s_j(a) - s_{j-1}(a) - s_j(\bar{a}) + s_{j-1}(\bar{a})) \\ &= \sum_{j=1}^{m} \ell_j (s_j(a) - s_j(\bar{a})) - \sum_{j=1}^{m} \ell_j (s_{j-1}(a) - s_{j-1}(\bar{a})) \\ &= \sum_{j=1}^{m} (\ell_j - \ell_{j+1})(s_j(a) - s_j(\bar{a})) \geqslant 0. \qquad \square \end{aligned}$$

A pattern a is said to be *dominant* if there does not exist any (feasible) pattern $\bar{a}$, $\bar{a} \neq a$, with $a <_d \bar{a}$. If ℓ and L are fixed then a pattern a can easily be identified to be dominant. Let

$$\delta(a) := \min\{\ell_m, \ell_{i-1} - \ell_i \colon a_i > 0, \ i \in I \setminus \{1\}\}.$$

Assertion 3. *The pattern a is dominant if and only if $L - \ell^{\mathrm{T}} a < \delta(a)$.*

Proof. "$\Rightarrow$" Suppose $L - \ell^{\mathrm{T}} a \geqslant \delta(a)$. If $\delta(a) = \ell_m$ then $a + e^m >_d a$. Otherwise there exists $i \in I \setminus \{m\}$ with $a_{i+1} > 0$ and $\ell^{\mathrm{T}} a + \ell_i - \ell_{i+1} \leqslant L$. Since the pattern $a + e^i - e^{i+1}$ dominates a, a cannot be dominant.

"$\Longleftarrow$" If a is not dominant then there exists a pattern $\bar{a}$, $\bar{a} \neq a$, with $a <_d \bar{a}$. The following two-step procedure can be applied at least once:

(i) Let j be the index determined by $\bar{a}_i = a_i$ for $i = 1, \ldots, j-1$ and $\bar{a}_j > a_j$.
(ii) If $j = m$ then one has $\ell^{\mathrm{T}} \bar{a} - \ell^{\mathrm{T}} a = \ell_m(\bar{a}_m - a_m) \geqslant \delta(a)$ – stop.
Otherwise a pattern $\tilde{a}$ is defined by $\tilde{a}_i := \bar{a}_i - 1$ for $i = j$, $\tilde{a}_i := \bar{a}_i + 1$ for $i = j + 1$, $\tilde{a}_i := \bar{a}_i$ otherwise.

Then

$$\ell^{\mathrm{T}} \bar{a} - \ell^{\mathrm{T}} \tilde{a} = \ell_j - \ell_{j+1} \quad \text{and} \quad \tilde{a}_{j+1} \geqslant 1. \tag{5}$$

Furthermore, $a <_d \tilde{a}$ since $s_i(\tilde{a}) = s_i(\bar{a})$ for $i \neq j$ and $s_j(\tilde{a}) = s_j(\bar{a}) - 1$.

Now, if $\tilde{a} = a$ then, because of (5), one has $\ell^{\mathrm{T}} \bar{a} - \ell^{\mathrm{T}} a \geqslant \delta(a)$ and hence, $L - \ell^{\mathrm{T}} a \geqslant \delta(a)$. Otherwise the procedure has to be repeated with $\tilde{a}$ instead of $\bar{a}$. □

Let $J_d \subset \{1, \ldots, n\}$ denote the index-set of dominant cutting patterns in A. Then:

Assertion 4. $(\ell, L) \in \mathbb{Z}_+^{m+1}$ *fulfils the first inequality system in* (4) *if and only if* (ℓ, L) *is feasible for*

$$\ell^{\mathrm{T}} a^j \leqslant L \quad \mathit{for}\ j \in J_d.$$

Proof. Since for any maximal cutting pattern a^j there exists a column a^k with $k \in J_d$ and $a^j <_d a^k$, $\ell^{\mathrm{T}} a^j \leqslant \ell^{\mathrm{T}} a^k \leqslant L$ follows. □

Furthermore, a maximal pattern a is said to be *non-dominant* if there does not exist any maximal pattern $\bar{a}$, $\bar{a} \neq a$, with $\bar{a} <_d a$. Non-dominant patterns can be identified similar to Assertion 3. Let $J_{nd} \subset \{1, \ldots, n\}$ be the index-set of non-dominant cutting patterns in A. Then:

Assertion 5. $(\ell, L) \in \mathbb{Z}_+^{m+1}$ *fulfils the second inequality system in* (4) *if and only if* (ℓ, L) *is feasible for*

$$\ell^{\mathrm{T}} (a^j + e^m) \geqslant L + 1 \quad \mathit{for}\ j \in J_{nd}.$$

Proof. Since for any maximal cutting pattern a^j there exists a column a^k with $k \in J_{nd}$ and $a^j >_d a^k$, $\ell^{\mathrm{T}} a^j \geqslant \ell^{\mathrm{T}} a^k \geqslant L + 1 - \ell_m$ follows. □

In order to illustrate the usage of dominance let us consider the instance

$$E_G := (5;\ 10000;\ 5000, 3750, 3250, 3001, 2000;\ 1, 1, 1, 1, 2) \tag{6}$$

presented in [6] with $\Delta(E_G) = \frac{16}{15} = 1.0667$. The instance E_G has 41 feasible patterns. Among them are 20 maximal, seven dominant and five non-dominant patterns. Hence, the class $K(E_G)$ of equivalent instances can be described as solution set of the system of inequalities:

$$\max\{2\ell_1, \ell_1 + 2\ell_5, 2\ell_2 + \ell_5, \ell_2 + 2\ell_4, \ell_2 + 3\ell_5, 3\ell_3, 5\ell_5\} \leqslant L,$$

$$\min\{\ell_1 + \ell_4 + \ell_5, \ell_1 + 3\ell_5, 2\ell_4 + 2\ell_5, \ell_4 + 4\ell_5, 6\ell_5\} \geqslant L + 1,$$

$$\ell_i \geqslant \ell_{i+1} + 1, \ i = 1, \ldots, 4, \quad \ell_5 \geqslant 1, \quad \ell_i \in \mathbb{Z}, \ i = 1, \ldots, 5.$$

Since $L = 60$, $\ell = (30, 22, 20, 19, 12)^{\mathrm{T}}$ is such a solution, E_G is equivalent to

$$E'_G := (5;\ 60;\ 30, 22, 20, 19, 12;\ 1, 1, 1, 1, 2).$$

3. Divisible case

In 1994, Nica [11] proposed a first infinite family of non-equivalent non-IRUP instances of the CSP which belongs to $\mathscr{D}$. For the sake of completeness his result is repeated here.

Let $m \geqslant 3$ and let integers $k_1, \ldots, k_m$ be given such that $2 \leqslant k_1 < \cdots < k_m$. The length L of the stock material is defined to be $\prod_{i=1}^{m} k_i$. Accordingly to the divisible case, the length ℓ_i of the ith piece equals L/k_i $(i \in I)$.

Using $k = (k_1, \ldots, k_m)^{\mathrm{T}}$, instances $E = E(k)$, proposed in [11], are as follows:

$$E(k) = \left(m;\ L;\ \frac{L}{k_1}, \ldots, \frac{L}{k_m};\ k_1 - 1, \ldots, k_{m-1} - 1, \left\lfloor k_m \sum_{i=1}^{m-1} \frac{1}{k_i} \right\rfloor \right).$$

Note, the instance E_D defined in (3) belongs to this family with $k = (3, 4, 11)^{\mathrm{T}}$.

Proposition 1 (Nica [11]). *Let $k \in \mathbb{Z}^m$ be given such that $2 \leqslant k_1 < \cdots < k_{m-1}$, $k_m + 1 = \prod_{i=1}^{m-1} k_i$ and all k_i are pairwise relatively prime. If $1 \leqslant b_m < k_m$ and if the order demands $b_i = k_i - 1$, $i = 1, \ldots, m-1$, cannot be cut using less than $m-1$ cutting patterns, then $E(k) \notin \mathscr{M}^*$.*

Proof. Because of construction one has

$$z_c(E(k)) = \sum_{i=1}^{m} \frac{b_i}{k_i} = \sum_{i=1}^{m-1} \frac{b_i}{k_i} + \frac{b_m}{k_m} \leqslant \sum_{i=1}^{m-1} \frac{k_i - 1}{k_i} + \frac{k_m \sum_{i=1}^{m-1} 1/k_i}{k_m} = m - 1$$

and $z_c(E(k)) + 1/k_m > m - 1$. Since the parameters k_i are assumed to be pairwise relatively prime there does not exist any non-negative integer vector $a = (a_1, \ldots, a_m)^{\mathrm{T}}$ with

$$\sum_{i=1}^{m} \frac{a_i}{k_i} \geqslant 1 \quad \text{and} \quad a_i \leqslant b_i, \ i = 1, \ldots, m.$$

This means, there does not exist any proper and trim-less pattern.

If $a=(a_1,\dots,a_m)^{\mathrm{T}}$ represents a maximal pattern then

$$a_m = \left\lfloor \left(L-\sum_{i=1}^{m-1}\ell_i a_i\right)\Big/\ell_m \right\rfloor = \left\lfloor k_m\left(1-\sum_{i=1}^{m-1}\frac{a_i}{k_i}\right)\right\rfloor$$

$$= \left\lfloor k_m-(1+k_m)\sum_{i=1}^{m-1}\frac{a_i}{k_i}+\sum_{i=1}^{m-1}\frac{a_i}{k_i}\right\rfloor = k_m-(1+k_m)\sum_{i=1}^{m-1}\frac{a_i}{k_i},$$

since $k_m+1=\prod_{i=1}^{m-1}k_i$ and $\sum_{i=1}^{m}\frac{a_i}{k_i}<1$.

If a set of $m-1$ cutting patterns $a^j=(a_{1j},\dots,a_{mj})^{\mathrm{T}}$, $j\in J_o$, $|J_o|=m-1$ is considered, fulfilling $\sum_{j\in J_o}a_{ij}=b_i=k_i-1$, $i=1,\dots,m-1$, then one gets

$$\sum_{j\in J_o}a_{mj}\leqslant (m-1)k_m-(1+k_m)\sum_{i=1}^{m-1}\frac{1}{k_i}\sum_{j\in J_o}a_{ij}=(m-1)k_m-(1+k_m)\sum_{i=1}^{m-1}\frac{k_i-1}{k_i}$$

$$=(m-1)k_m+(1+k_m)\sum_{i=1}^{m-1}\frac{1}{k_i}-(1+k_m)(m-1)$$

$$=(1+k_m)\sum_{i=1}^{m-1}\frac{1}{k_i}-(m-1).$$

Because of

$$b_m=\left\lfloor k_m\sum_{i=1}^{m-1}\frac{1}{k_i}\right\rfloor=(k_m+1)\sum_{i=1}^{m-1}\frac{1}{k_i}-\left\lceil\sum_{i=1}^{m-1}\frac{1}{k_i}\right\rceil$$

one obtains

$$b_m-\sum_{j\in J_o}a_{mj}\geqslant m-1-\left\lceil\sum_{i=1}^{m-1}\frac{1}{k_i}\right\rceil\geqslant 1$$

since $m\geqslant 3$ and $k_1\geqslant 2$ are assumed. Hence, at least one more piece of length ℓ_m is ordered as can be cut with $m-1$ cutting patterns. □

Note, the gap

$$\Delta(E(k))=1+\frac{1}{k_m}\left(\left\lceil\sum_{i=1}^{m-1}\frac{1}{k_i}\right\rceil-\sum_{i=1}^{m-1}\frac{1}{k_i}\right)$$

is asymptotically bounded by $1+1/k_m$ and tends to 1 if k_m is increased.

Next, by means of an example it is shown that non-IRUP instances can also be obtained in the divisible case if some of the assumptions in Proposition 1 are not fulfilled. For this, let us consider the instance

$$E_1=(6;\ 4620;\ 1540,1155,924,660,420,77;\ 2,3,4,6,10,1)$$

with $k=(3,4,5,7,11,60)^{\mathrm{T}}$. Here $k_m+1\neq\prod_{i=1}^{m-1}k_i$ and the k_i are not pairwise relatively prime. Using the proper relaxation bound (cf. [12]) $E_1\notin\mathscr{M}^*$ can be verified. Moreover,

$\ell_1+\ell_2+2l_3+\ell_6=L$. Hence, separating this pattern a new instance

$$E_2=(5;\ 4620;\ 1540,1155,924,660,420;\ 1,2,2,6,10)$$

can be derived from E_1 with $E_2\notin\mathscr{M}^*$.

Now another family of instances of $\mathscr{D}$ with $m=3$ is considered. Let p be any positive integer and let $k=(3p,3p+1,9p+2)^{\mathrm{T}}$. L is defined to be the smallest common multiple of the k_i (or a multiple of it). For abbreviation, let $q=3p$. Then the family of instances

$$E(p)=\left(3;\ L;\ \frac{L}{3p},\frac{L}{3p+1},\frac{L}{9p+2};\ 3p-1,3p,6\right)$$
$$=\left(3;\ L;\ \frac{L}{q},\frac{L}{q+1},\frac{L}{3q+2};\ q-1,q,6\right)$$
$$q=3p,\ \ p\geqslant 1,\ \text{integer},$$

will be investigated. Notice, E_D defined in (3) belongs to this family with $p=1$.

Proposition 2. *For any integer $p\geqslant 1$, instance $E(p)$ does not belong to $\mathscr{M}^*$.*

Proof. Since $E_D=E(1)$, $p>1$ can be assumed. Because of $E(p)\in\mathscr{D}$ one has

$$z_c=\frac{q-1}{q}+\frac{q}{q+1}+\frac{6}{3q+2}=2-\frac{q+2}{Q},$$

where $Q=q(q+1)(3q+2)$. Because of $z_c\leqslant 2$, the instance $E(p)$ would belong to $\mathscr{M}^*$ if and only if there exist any proper pattern $a=(a_1,a_2,a_3)^{\mathrm{T}}$ with

$$\frac{a_1}{q}+\frac{a_2}{q+1}+\frac{a_3}{3q+2}\geqslant z_c-1=1-\frac{q+2}{Q}. \tag{7}$$

It will be shown that such a pattern cannot occur. Since in the divisible case $\sum_{i=1}^{m}a_i/k_i\leqslant 1$ for any feasible pattern a it follows:

$$(q+1)a_1+qa_2\leqslant q^2+q-\frac{q^2+q}{3q+2}a_3.$$

Since

$$\frac{q^2+q}{3q+2}=\frac{q}{3}+\frac{1}{9}-\frac{2}{9(3q+2)}>\frac{q}{3} \tag{8}$$

the following inequality must be fulfilled:

$$(q+1)a_1+qa_2<q^2+q-\frac{q}{3}a_3. \tag{9}$$

Because of (7) and (8) one has

$$(q+1)a_1+qa_2\geqslant q^2+q-\frac{q^2+q}{3q+2}a_3-\frac{q+2}{3q+2}$$
$$>q^2+q-\left(\frac{q}{3}+\frac{1}{9}\right)a_3-\frac{q+2}{3q+2}.$$

Furthermore, with

$$\frac{q+2}{3q+2} = \frac{1}{3} + \frac{4}{3(3q+2)}$$

it follows that the pattern a has to fulfil the inequality

$$(q+1)a_1 + qa_2 > q^2 + q - \left(\frac{q}{3} + \frac{1}{9}\right) a_3 - \frac{1}{3} - \frac{4}{3(3q+2)}. \tag{10}$$

Summarizing (9) and (10), if $E(p) \in \mathscr{M}^*$ then there must exist a proper pattern a with

$$(q+1)a_1 + qa_2 - q^2 - q \in \left(-\left(\frac{q}{3} + \frac{1}{9}\right) a_3 - \frac{1}{3} - \frac{4}{3(3q+2)}, -\frac{q}{3} a_3\right). \tag{11}$$

Note, the left-hand side is integral. In case $a_3 = 6$, the interval has the form

$$\left(-2q - 1 - \frac{4}{3(3q+2)}, -2q\right)$$

which contains only the single integer $-2q-1$. The corresponding equation

$$(q+1)a_1 + qa_2 = q^2 + q - 2q - 1 = q^2 - q - 1$$

can be transformed in

$$a_2 = -a_1 + q - 1 - \frac{1}{q}(a_1 + 1)$$

and for $a_1 \leqslant q-1$, a_1 integer, it is found there exists no non-negative integer value a_2.

In the other case, $a_3 \leqslant 5$, because of $q = 3p$, the interval in formula (11) can be written in the form

$$\left(-pa_3 - \frac{a_3}{9} - \frac{1}{3} - \frac{4}{3(9p+2)},\ -pa_3\right).$$

For $p \geqslant 2$ the interval size is less than $\frac{5}{9} + \frac{1}{3} + \frac{1}{15} < 1$. Hence, there is no integer contained in the interval. Therefore, no such pattern can exist. $\square$

4. Non-divisible case

The family

$$E(t) = (7;\ 51 + 3t;\ 23 + t, 19 + t, 17 + t, 16 + t, 15 + t, 14 + t, 13 + t;$$

$$1, 2, 2, 1, 1, 1, 1),$$

$$t \in T := \{t \in \mathbb{Q}\colon t > -13,\ t \neq -12, -11, -9\}$$

is considered next. Note, this family does not contain only non-equivalent instances. For example, for $t > -1$ all instances $E(t)$ are equivalent and $k = (2, 2, 3, 3, 3, 3, 3)^{\mathrm{T}}$.

Proposition 3. *All instances $E(t)$, $t \in T$, do not belong to $\mathscr{M}^*$.*

Proof. $z_c(E(t))=3$ and $z^*(E(t))=4$ for $t\in T$ is shown.

Because of $\ell^{\mathrm{T}}b=153+9t$ one has $z_c(E(t))\geqslant 3$. On the other hand, $z_c(E(t))\leqslant 3$ since $\frac{1}{2}(e^1+2e^6)$, $\frac{1}{2}(e^2+2e^4)$, $\frac{1}{2}(3e^3)$, $\frac{1}{2}(e^2+e^3+e^5)$, $\frac{1}{2}(e^1+e^5+e^7)$, $\frac{1}{2}(2e^2+e^7)$, forms a feasible solution. Suppose $z^*(E(t))=3$. Since $\ell^{\mathrm{T}}b=3L$ the optimal solution must contain only proper patterns a with $\ell^{\mathrm{T}}a=L$. The pattern with $a_1=1$ is considered. Then, because of $L-\ell_1=28+2t$, $\ell_1+\ell_5+\ell_7$ is the only trim-less proper pattern. But now there does not exist any proper trim-less pattern with $a_2\geqslant 1$ since ℓ_5 is already used. □

Remark. The instance $E(t)$ with $t=-10$ is a non-IRUP instance with only integer data. Here $L=21$. But there are even smaller stock length L for which a non-IRUP instances exists. The instances (4; 18; 9,7,6,4; 1,1,2,2) and (5; 16; 10,8,7,3,2; 1,1,1,1,2) are the smallest known so far.

Using the substitution $p/q:=t+13$ for $t\in(-13,-12)$ with $0<p<q,\ p,q\in\mathbb{Z}$, a further family of instances can be obtained from $E(t)$ by removing the smallest piece:

$$E(p,q)=(6;\ 12q+3p;\ 10q+p,6q+p,4q+p,3q+p,2q+p,q+p;$$
$$1,2,2,1,1,1),\quad 0<p<q,\ p,q\in\mathbb{Z}. \tag{12}$$

Proposition 4. *Let $p/q\in(0,1)$, $p,q\in\mathbb{Z}_+$, then the instance $E(p,q)$ defined in* (12) *does not belong to $\mathscr{M}^*$.*

The proof is similar to that of Proposition 3. Moreover, in case of $p>1$ the length of the smallest piece can be reduced to $q+1$ without violating the non-IRUP property. That means, for $r\in\{1,\ldots,p-1\}$,

$$E_r(p,q)=(6;\ 12q+3p;\ 10q+p,6q+p,4q+p,3q+p,2q+p,q+r;$$
$$1,2,2,1,1,1)\notin\mathscr{M}^*.$$

Next, the family

$$E(p)=(5;\ (2p+1)^2+1;\ 2p^2+2p+2,2p^2+2p+1,2p^2+2,2p+2,2p;$$
$$1,1,1,1,p),\quad p>1,\ p\in\mathbb{Z} \tag{13}$$

of non-equivalent CSP instances is considered.

Proposition 5. *The instances $E(p)$, $p=2,3,\ldots,$ defined in* (13) *do not belong to $\mathscr{M}^*$.*

Proof. Because of

$$\frac{1}{2}(2e^2)+\frac{p}{p+1}(e^1+(p+1)e^5)+\frac{p-1}{Q}(e^1+e^3)$$
$$+\frac{p^2+p+1}{Q}(2e^3+e^4)+\frac{p+2}{Q}(e^1+pe^4)=b \tag{14}$$

where $Q=(p+1)(2p+1)$, $z_c(E(p))\leqslant 2-(p-1)/(2Q)$ holds. Furthermore, because of $\ell^{\mathrm{T}}b=8p^2+6p+7=2L-(2p-3)$ the total trim-loss has to be equal to $(2p-3)$ if

$z^*(E(p))=2$ is assumed. But the best proper pattern a with $a_1=1$ is $a=e^1+e^4+(p-1)e^5$ which has a trim-loss of $2p-2$ units. Hence, $z^*(E(p))>2$ must hold. $\square$

Notice, $\lim_{p\to\infty} z_c(E(p))=2$ which corresponds to $\lim_{p\to\infty} k_5=\lim_{p\to\infty} 2p+2=\infty$. Moreover, since the gaps are asymptotically bounded by $1+(2p-3)/L$ the gaps tend to 1 for $p\to\infty$. The maximal gap for this family is $\frac{29}{28}$ obtained for $p=3$.

The following procedure leads to further non-equivalent non-IRUP instances if $p\geqslant 5$: if pieces of length $\ell_5=2p$ are composed to larger ones then no better integer solution can arise. If additionally the value of the continuous relaxation does not increase then non-IRUP instances will be constructed. Let $p_i\in\mathbb{Z}$, $p_i\geqslant 2$, and $\beta_i\in\mathbb{Z}$, $\beta_i>0$, $i=1,\dots,q$ $(q\geqslant 2)$, such that $p\geqslant\sum_{i=1}^{q}\beta_i p_i$ and $(p+1)/p_i\in\mathbb{Z}$ for $i=1,\dots,q$. Without loss of generality, let $p_1>\cdots>p_q$. Then a non-equivalent instance $E'(p)$ derived from $E(p)$ is as follows in case of $p=\sum_{i=1}^{q}\beta_i p_i$:

$$E'(p)=(m';\ L;\ \ell';\ b')$$

$$:=(4+q;\ L;\ \ell_1,\ell_2,\ell_3,p_1\ell_5,\dots,p_q\ell_5,\ell_4;\ 1,1,1,\beta_1,\dots,\beta_q,1).$$

Using $\beta_i p_i/(p+1)$ times the pattern $\ell'_1+((p+1)/p_i)\ell'_{3+i}=\ell_1+((p+1)/p_i)(p_i\ell_5)$, $i=1,\dots,q$, instead of $p/(p+1)$-times the pattern $(1,0,0,0,p+1)^{\mathrm{T}}$ in (14), a feasible solution of the LP relaxation with the same objective function value can be obtained.

For example, let $p=5$. Then from $E(5)=(5;\ 122;\ 62,61,52,12,10;\ 1,1,1,1,5)$ using $q=2$, $p_1=3$, $p_2=2$ and $\beta_1=\beta_2=1$, $E'(5)=(6;\ 122;\ 62,61,52,30,20,12;\ 1,1,1,1,1,1)$ is obtained. Note, $\Delta E(5)=\Delta E'(5)=\frac{34}{33}=1.03\overline{03}$.

Furthermore, if $p\geqslant 5$, odd, $p+1\neq 2^r$ with $r\in\mathbb{Z}$ and $p=\sum_{i=1}^{q}\beta_i p_i$ then, because of construction, the length $\ell_3=\ell'_3$ now can be reduced to $2p^2$ since no new proper pattern becomes feasible. On the other hand, new non-proper patterns can lead to a smaller optimal value of the continuous relaxation. In the example $E''(5)=(6;\ 122;\ 62,61,50,30,20,12;\ 1,1,1,1,1,1)$ is obtained with $\Delta E''(5)=\frac{37}{35}=1.0571$.

Next a modification of $E''(5)$ is considered. Let $E(b)=(6;\ 122;\ 62,61,50,30,20,12;\ 3,3,3,3,3,4)$. Here one has $z_c(E(b))=\frac{207}{35}=5.9143$ and $z^*(E(b))=7$. Thus $E(b)\notin\mathscr{M}^*$. (The verification of $z^*(E(b))=7$ was done using the cutting plane algorithm proposed in [18].) Note, $\Delta(E(b))=\frac{38}{35}=1.0857$. Hence, $E(b-a)\notin\mathscr{M}^*$ for any proper pattern a of E with $z_c(E(b-a))\leqslant 5$. If $a\in\{e^1+2e^4,e^1+3e^5,2e^2,2e^3+e^5\}$ the same gap arises; for $a\in\{e^1+e^4+2e^6,e^2+e^3,2e^3+e^6\}$ smaller gaps occur. The corresponding residual instance of $E(b)$ is $E=(6;\ 122;\ 62,61,50,30,20,12;\ 2,1,1,1,2,4)$. Residual (cf. [15]) means there exists an optimal solution x^c of the LP relaxation with components less than 1 so that x^c cannot be used for a further reduction of the order demands.

Non-IRUP instances can also be found for special classes of CSPs. Let us consider instances $E=(m;\ L;\ \ell;\ b)$ with the property $k_i\in\{1,k\}$ for $i\in I$ where $k\in\mathbb{Z}$, $k\geqslant 2$, and $k_i=\lfloor L/\ell_i\rfloor$. For short such instances are referred as $(1,k)$-instances. It is known, all $(1,k)$-instances belong to $\mathscr{M}^*$ for $k\in\{2,3\}$. But for any $k\geqslant 4$ there exist non-IRUP $(1,k)$-instances. Such instances for $k=4$ and 5 are

$$k=4\colon\ E_4=(9;\ 415;\ 239,235,233,93,91,90,89,88,87;\ 1,1,1,1,1,1,1,1,1) \tag{15}$$

and

$$k=5\colon E_5=(9;\ 465;\ 289,285,283,93,91,90,89,88,87;\ 1,1,1,1,1,1,1,1,1). \tag{16}$$

A family of non-IRUP $(1,k)$-instances with $k \geqslant 6$ is the following:

$$\begin{aligned} E_k = (8;\ & 10k^2+11k;\ 10k^2-19k-15, 10k+11, 10k+7, 10k+5, 10k+4, \\ & 10k+3, 10k+2, 10k+1;\ 3,1,2,2,1,1,1,1), \quad k \geqslant 6. \end{aligned} \tag{17}$$

Proposition 6. *The instances E_k, $k \geqslant 4$, defined in* (15)–(17), *do not belong to $\mathscr{M}^*$.*

Proof. The statement for $k=4$ and 5 can easily be verified. Let $k \geqslant 6$. Obviously, the instance E_k is a $(1,k)$-instance. Since $\ell_1 + 4\ell_8 > L$ at most three smaller pieces can be contained in a pattern a with $a_1 = 1$. Because of $b_1 = 3$, $z^*(E_k) \geqslant 3$ follows. The reduced instance E_k' obtained from E_k by omitting the three pieces of length ℓ_1 and by setting the stock length to $L - \ell_1$ corresponds to the instance $E(t)$ with $t = 10k - 12$ considered in Proposition 3. Therefore, $E_k' \notin \mathscr{M}^*$ and hence, since $z_c(E_k) = 3$, $E_k \notin \mathscr{M}^*$. $\square$

Remember, the instance $E_k' = E(t)$ with $t = 10k - 12$ has special structure, namely one has $k_i \in \{2,3\}$ for any i. Generally, an instance $E = (m;\ L;\ \ell;\ b)$ is said to be a $(k,k+1)$-instance if $k_i = \lfloor L/\ell_i \rfloor \in \{k,k+1\}$ for any i ($k \in \mathbb{Z}$, $k > 0$).

The proof of Proposition 6 suggests the following procedure to construct non-IRUP $(1,k)$-instances. Let a non-IRUP $(p,p+1)$-instance $E=(m;\ L;\ \ell;\ b)$ be given. (Then $p \geqslant 2$.) In order to obtain a non-IRUP $(1,k)$-instance at first an instance $E'=(m;\ L';\ \ell';\ b)$ is constructed as follows. The piece lengths of E are increased by t (t sufficiently large), i.e. $\ell_i' := \ell_i + t$, $i \in I$, and $L' := L + (p+1)t$. Then, any feasible (with respect to E) pattern a with $e^{\mathrm{T}} a \leqslant p+1$ (where $e = (1,\ldots,1)^{\mathrm{T}}$) is also feasible for E', and any non-feasible (with respect to E) pattern a with $e^{\mathrm{T}} a \geqslant p+1$ is also non-feasible for E'. If t tends to infinity then L'/ℓ_i' tends to $p+1$ for $i \in I$. If $\lceil z_c(E) \rceil$ pieces of length ℓ_0' (ℓ_0' sufficiently large) are added and the stock length is increased by ℓ_0', then

$$k \leqslant \frac{L' + \ell_0'}{\ell_i'} < k+1, \quad i \in I,$$

can be reached for $k \geqslant 2p+2$. Hence, a $(1,k)$-instance with $m+1$ piece types is obtained.

Let us now consider the set of $(k,k+1)$-instances with $k \in \mathbb{Z}$, $k > 0$. For $k=1$ all (1,2)-instances belong to $\mathscr{M}^*$, but for any $k \geqslant 2$ there exist non-IRUP $(k,k+1)$-instances as will be shown next. Let $\lambda \geqslant \max\{3k^3 - k^2 - 3k, 3k^2 + 3k + 1\}$ and

$$\begin{aligned} E_k = (9;\ & (k+1)(\lambda+3k);\ \lambda+3k^2+2k, \lambda+2k^2+k, \lambda+6k, \lambda+1, \lambda+k+2, \\ & \lambda+3k-3, \lambda, \lambda+k, \lambda+3k;\ 1,1,1,k-1,k-1,k-1,1,1,1), \quad k \geqslant 2. \end{aligned} \tag{18}$$

Then E_k is a $(k,k+1)$-instance. Note, in order to make more clear the inherent structure the piece lengths are not sorted in decreasing order.

Proposition 7. *The instances E_k, $k \geqslant 2$, defined in* (18), *do not belong to $\mathscr{M}^*$.*

Proof. Since $\ell^{\mathrm{T}}b=3L$ for any k, $z_c(E_k) \geqslant 3$. Since $(1/k)(e^1+(k-1)e^7+e^8)$, $(1/k)(e^2+(k-1)e^8+e^9)$, $(1/k)(e^3+e^7+(k-1)e^9)$, $((k-1)/k)(e^1+ke^4)$, $((k-1)/k)(e^2+ke^5)$, $((k-1)/k)(e^3+ke^6)$, forms a feasible solution, $z_c(E_k)=3$.

If $z^*(E_k)=3$ is supposed then no trim-loss must occur. Patterns a with $a_1=1$ are considered. Let $k>2$. Since $k\ell_1<L$ and $\ell_1>\ell_i$ for $i\neq 1$ a trim-less pattern must contain $k+1$ pieces. Because of construction, $\ell_1+k\ell_4=L$ but $b_4=k-1$ so this pattern is not proper. Furthermore, $\ell_1+(k-1)\ell_7+\ell_8=L$ but again this pattern is not proper. Since $\ell_7<\ell_4<\ell_8<\ell_i$ for all other i there is no trim-less pattern a with $a_1=1$. If $k=2$, $\ell_1+\ell_7+\ell_8=L$ yields the only proper pattern with $a_1=1$. But if it is used then the lengths ℓ_2 and ℓ_3 cannot be obtained without waste. Hence, $z^*(E_k)>3$ follows. □

5. Further constructions

Let $p\in\mathbb{Z}$ with $p\geqslant 2$ and let

$$E'=(m';\ L';\ \ell'_1,\ldots,\ell'_{m'};\ b'_1,\ldots,b'_{m'})$$

be an instance not belonging to $\mathscr{M}^*$ with $L',\ell'_1,\ldots,\ell'_{m'}\in\mathbb{Q}$, $L'>\ell'_1>\cdots>\ell'_{m'}>0$, $b'_1,\ldots,b'_{m'}\geqslant 1$. Moreover, let $q\in\mathbb{Q}$ with $q\geqslant 2/\ell_{m'}$. Then the instance E is defined by

$$\begin{aligned} E &= (m;\ L;\ \ell_1,\ldots,\ell_m;\ b_1,\ldots,b_m)\\ &:= (m'+3;\ p(qL'+1);\ L-qL',qL'+1,qL'+2-q\ell'_{m'},q\ell'_1,\ldots,q\ell'_{m'};\\ &\qquad \lceil z_c(E')\rceil, p-1,1,b'_1,\ldots,b'_{m'}). \end{aligned}$$

Proposition 8. *If for any $a'\in\mathbb{Z}^m_+$ with $\ell'^{\mathrm{T}}a'>L'$ and $a'\leqslant b'$*

$$q\left(\sum_{i=1}^{m'}\ell'_i a'_i - L'\right)>1$$

then E does not belong to $\mathscr{M}^*$ too.

Proof. At first $\lceil z_c(E)\rceil \leqslant \lceil z_c(E')\rceil+1$ is proved.

Let us consider an optimal solution of the LP relaxation of E' characterized by the set $\{a'^j\}_{j\in J^*}$ of feasible patterns and corresponding coefficients x_j, $j\in J^*$, i.e. $\sum_{j\in J^*}a'_{ij}x_j=b'_i$, $i=1,\ldots,m'$, $z_c(E')=\sum_{j\in J^*}x_j$ and $x_j>0$, $j\in J^*$.

Now to any a'^j $(j\in J^*)$ an m-dimensional pattern a^j is assigned as follows:

$$a_{1j}=1,\quad a_{ij}=0,\ i=2,3,\quad a_{ij}=a'_{i-3,j},\ i=4,\ldots,m.$$

The patterns a^j are feasible with respect to E. Then, $\sum_{j\in J^*}a_{ij}x_j=b_i$, $i=4,\ldots,m$.

Using $(\lceil z_c(E')\rceil - z_c(E'))$-times the feasible pattern $\bar{a}^1 := e^1 + e^3$ one has $\sum_{j\in J^*} a_{1j}x_j + (\lceil z_c(E')\rceil - z_c(E')) = b_1$.

Supplementing $(p-1)/p$-times the pattern $\bar{a}^2 := pe^2$ and $(1-\lceil z_c(E')\rceil + z_c(E'))/p$-times the pattern $\bar{a}^3 := pe^3$, a feasible solution of the continuous relaxation of E is found (which is not necessarily optimal). Hence,

$$\begin{aligned} z_c(E) &\leqslant z_c(E') + (\lceil z_c(E')\rceil - z_c(E')) + (p-1)/p + (1 - \lceil z_c(E')\rceil + z_c(E'))/p \\ &= \lceil z_c(E')\rceil + 1 - (\lceil z_c(E')\rceil - z_c(E'))/p \leqslant \lceil z_c(E')\rceil + 1. \end{aligned}$$

Next $z^*(E) > \lceil z_c(E')\rceil + 1$ is shown. Because of $\ell_1 + \ell_2 > L$, $\lceil z_c(E')\rceil$ patterns $\bar{a}^j$ with $\bar{a}_{1j} = 1$ and $\bar{a}_{2j} = 0$ are needed to cut the b_1 pieces of length ℓ_1. If an integer solution for E is supposed with $\lceil z_c(E')\rceil + 1$ patterns then all $p-1$ pieces of length ℓ_2 must be in one pattern $\bar{a}$ with $\bar{a}_1 = 0$. Then the total length in $\bar{a}$ not covered with ℓ_2-pieces equals $qL' + 1$.

Since $E' \notin \mathscr{M}^*$ all the pieces with lengths $\ell_4, \dots, \ell_m$ cannot be cut with these $\lceil z_c(E')\rceil$ patterns $\bar{a}^j$. At least one piece (with length $\geqslant q\ell'_{m'}$) remains unpacked. Since

$$\ell_3 + q\ell'_{m'} = qL' + 2 - q\ell'_{m'} + q\ell'_{m'} = qL' + 2 > qL' + 1$$

this piece and the piece of length ℓ_3 cannot be cut in pattern $\bar{a}$. Hence, one more pattern is needed, that means $z^*(E) > \lceil z_c(E')\rceil + 1$. □

Let us consider the instance $E = E'_G = (5;\ 60;\ 30, 22, 20, 19, 12;\ 1, 1, 1, 1, 2)$ which is equivalent to that proposed by Gau ([6], cf. (6)). Here one has $\Delta(E) = \frac{16}{15}$. If the procedure corresponding to Proposition 8 is applied with $p = 2$ and $q = 2$, the instance

$$E' = (8;\ 242;\ 122, 121, 98, 60, 44, 40, 38, 24;\ 2, 1, 1, 1, 1, 1, 1, 2)$$

results. For this instance one gets $z_c(E') = \frac{26}{9}$ and $z^*(E') = 4$ so that $\Delta(E') = \frac{10}{9}$ is obtained. This is the largest gap found so far [13]. If this procedure is applied to E with $p = 3, 4, 5, 6$ and $q = 2$ then the gaps 1.0741, 1.0556, 1.0444 and 1.0394 result. Note, the repeated application of Proposition 8 is possible but does not succeed with larger gaps.

There are some other possibilities to obtain non-IRUP instances. Let an instance $E = (m; L; \ell; b)$ be given. If the order demand of the mth piece is increased by some units, say $\lambda \in \mathbb{Z}$, $\lambda > 0$, then three cases can occur for $E_\lambda = (m; L; \ell; b + \lambda e^m)$.

If $z_c(E_\lambda) > \lceil z_c(E)\rceil$ and $z^*(E_\lambda) = z^*(E)$ then $E \notin \mathscr{M}^*$.

If $z_c(E_\lambda) \leqslant \lceil z_c(E)\rceil$ and $z^*(E_\lambda) > z^*(E)$ then $E_\lambda \notin \mathscr{M}^*$.

Otherwise, E and E_λ have the same behaviour.

The instances considered in this paper are mostly residual instances. But, if the order demand b of a non-IRUP instance E is increased by a non-negative integer combination of patterns occurring in an LP solution of E then a non-IRUP instance is obtained very often. For example, if E_D defined in (3) is used then one gets

$$(3;\ 132;\ 44, 33, 12;\ 2 + 3\lambda_1, 3 + 4\lambda_2, 6 + 11\lambda_3) \notin \mathscr{M}^* \quad \text{for } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_+.$$

To verify such statements stronger relaxations of the CSP are needed as proposed in [12,18].

6. Concluding remarks

Within series of randomly generated test instances of the CSP non-IRUP instances occur relatively rarely, but these are the hard instances for exact solution approaches. In this paper families of non-IRUP instances are presented. Especially, such instances should be used to analyse the computational behaviour of exact solution methods in addition to the analysis of their average behaviour.

As a consequence of investigations with respect to the MIRUP-conjecture, efficient heuristics should not terminate without an integer solution worser than one unit as the LP bound rounded up.

Moreover, since the gaps tend to 1 for increasing parameters counter-examples for the MIRUP-conjecture, if such exist, have probably small ratios between stock material length and piece lengths.

There remain some open questions. Are there structural criteria to identify efficiently instances to be a non-IRUP instance? The MIRUP-conjecture still remains to be proved. Possibly the investigation of related optimization problems where also the difference between optimal value and LP bound is restricted, may lead to further results.

The investigations with respect to the modified round-up property are useful for developing efficient solution approaches. Moreover, our investigations may be helpful for finding a suitable theoretical approach in order to investigate the phenomenon that there is a very small gap between the optimal values of the integer problem and the LP relaxation.

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